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# Teichmüller Spaces of String Theory

**K. Bugajska**[1]

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## 1. INTRODUCTION

In string theory the fundamental objects are one dimensional and propagate in space-time $\mathbb{R}^{1,D}$. As a string propagates it sweeps out a Lorentzian world sheet $\Sigma^L$. Recall that the Teichmüller space $\mathscr{T}_{p,n}$ is the space of conformal structures on a topological surface $S$ of genus $p$, where two structures are equivalent if there is a conformal map between them which is homotopic to the identity (Bers, 1981). Physicists incorporate Teichmüller space into the theory by moving between the Euclidean and Lorentzian conventions for the world-sheet signature with lack of concern. So, with the theory of Riemann surface an important mathematical tool for string theory, "The string is described by immersion $X^\mu(\sigma^\alpha)$ of its (compact) two dimensional world sheet with coordinates $\sigma^\alpha$ into Euclidean space-time." In this paper we will not reject the Lorentzian signature, but just the-opposite, we will investigate the consequences of the fact that the signature of a metric of a string world sheet $\Sigma^L$ immersed into $\mathbb{R}^{1,D}$ is a Lorentzian one. We will show that for such a Lorentzian world sheet $\Sigma^L$ any "observer" (identified with a concrete unit timelike vector $e_0 \in \mathbb{R}^{1,D}$) "produces" some curve in an appropriate Teichmüller space $\mathscr{T}_{p,n}$ which we will call a $P$-line ($P$ for "physics"). Here $p$ denotes the genus of $\Sigma^L$, $p \geqslant 2$, and $n$ denotes the number of punctures. Next we prove that such a $P$-line has to be an infinite geodesic in the Teichmüller metric of $\mathscr{T}_{p,n}$.

In Section 2 we construct our $P$-line in two different ways: first, using a timelike (singular) vector field $\mathbf{v}_1$ on $\Sigma^L$ determined by $e_0 \in \mathbb{R}^{1,D}$ and second, using a pair of transverse measured foliations determined on $\Sigma^L$

[1]Department of Mathematics, York University, North York, Ontario, Canada M3J 1P3.



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also by $e_0$. Although the second method is much more elegant, the first one appears to be useful also, especially when we pass to a physical interpretation. The relation between singularities of the vector field $\mathbf{v}_1$ (and the Poincaré–Hopf theorem describing them) and zeros of holomorphic 1-form distributions determining a pair of transverse measured foliations (and the Riemann–Roch theorem describing them) are considered at the end of Section 2.

In Section 3 we investigate the so-called harmonic $P$-lines, i.e., the case when the immersion $X:\Sigma \to \mathbb{R}^{1+D}$ determined by the fixed vector $e_0 \in \mathbb{R}^{1,D}$ induces the harmonic homothetic Gauss map $g$: $\Sigma \to G_{2,D+1}$; here $\Sigma$ is the underlying manifold of the world sheet $\Sigma^L$.

In Section 4 we introduce Jenkins–Strebel differentials and give a very short description of some of their properties which we need later. We use the properties of $b$-boundary points of the Bers embedding of a Teichmüller space to show strong consequences follow from the fact that the quadratic differential related to the $P$-line is a Jenkins–Strebel one. In this case the endpoints of any $P$-line are represented by regular $b$-groups. Thus these endpoints describe the decomposition of our world sheet onto one or more Riemann surfaces which may be thought to have been obtained by cutting along an admissible system of Jordan curves (determined by the above mentioned quadratic differential) and then by gluing a punctured disc to each side of each cut. So if we can relate a physical object to the Lorentzian world sheet $\Sigma^L$ then this object has to be created (from other objects) and has to decay.

These considerations suggest that the set of "Lorentzian observers" $e_0$'s who can observe the same physical properties has to be discrete. More exactly, we obtain that a discrete, although infinite subgroup of the Lorentzian group $SO_+(1, D)$ is the symmetry of our theory. To see this all we need is the existence of a $P$-line determined by a Jenkins–Strebel (JS) differential with only one cylinder of each type (i.e., horizontal and vertical) whose heights and circumferences are equal to each other, respectively. This is the subject of Section 5.

In Section 6 we describe $P$-lines (which are geodesics in the Teichmüller metric) using the Weil–Petersson metric on $\mathscr{T}_{p,n}$. By introducing Fenchel–Nielsen coordinates we can show that $P$-lines corresponding to JS differentials with $3p-3$ cylinders can be related to a Hamiltonian system with respect to the Weil–Petersson–Kaehler symplectic form $\omega$ on $\mathscr{T}_{p,n}$.

In Bugajska (1990, 1991) we construct reductions of maximally unstable (associated to concrete spinor structures), holomorphic $SL(2, \mathbb{C})$ bundles (over Riemann surfaces lying on a $P$-line) to the $SU(2)$ group. Since on Riemann surface $G$-bundles with connection are equivalent to holomorphic $G^{\mathbb{C}}$ bundles with a reduction to $G$ and since $SU(2)^{\mathbb{C}} = SL(2, \mathbb{C})$, we obtain appropriate $SU(2)$ bundles with connection. We can interpret this result as a gauge field of weak interactions. So we obtain that these completely different approachers to our $P$-lines [described in Section 4 and Bugajska (1990, 1991)] yield the same physical situation, namely decay and creation of objects related to a world sheet $\Sigma^L$.

## 2. $P$-LINES AS TEICHMÜLLER GEODESICS

Let us assume that we have some one-dimensional object in $(1+D)$-dimensional Minkowski space-time $\mathbb{R}^1$. Let us assume that as this object propagates in $\mathbb{R}^{1,D}$ it sweeps out a Lorentzian world sheet $\Sigma^L$. Moreover, let us assume that this world sheet forms a connected, orientable manifold $\Sigma$ of genus $p \geqslant 2$. Now we can ask which Riemann surfaces (i.e., which complex structures) can be related to $\Sigma^L$, or equivalently, we can ask how we can describe $\Sigma^L$ in an appropriate Teichmüller space $\mathscr{T}_{p,n}$.

$\Sigma^L$, as a manifold, carries a $GL_+(2, \mathbb{R})$ structure, i.e., there exists a natural $GL_+(2, \mathbb{R})$ principal bundle $\xi_{GL_+}$ of oriented linear frames over its underlying manifold $\Sigma$. The immersion $X$ of $\Sigma$ into Minkowski space-time $\mathbb{R}^{1,D}$ determines a concrete reduction of the bundle $\xi_{GL_+}$ to the $SO(1, 1)$ principal bundle $\xi_s$ (a.e.), i.e.,

$$X: \quad \Sigma \to \mathbb{R}^{1,D} \Rightarrow \xi_{GL_+} \to \xi_s \tag{2.1}$$

Next, any complex structure on $\Sigma$ will be given by a reduction of the $\xi_{GL_+}$ principal bundle to the $GL(1, \mathbb{C})$ subgroup. Since $GL(1, \mathbb{C}) \cong \mathbb{C}^* \cong SO(2) \times \mathbb{R}^+$, we see that we have one-to-one correspondence between complex and conformal structures on $\Sigma$. To find which of them can be related to our world sheet, first let us consider the two-dimensional real vector space $\mathbb{R}^2$.

On $\mathbb{R}^2$ the family of oriented linear frames corresponds to elements of the group $GL_+(2, \mathbb{R})$. Moreover, each linear frame defines uniquely a Euclidean structure on $\mathbb{R}^2$ as well as a unique Lorentzian structure on it. Let us fix one such frame, say $\{\mathbf{v}_1, \mathbf{v}_2\}$, and let us consider the Lorentzian structure $\mathbb{R}^{1,1}$ determined by this.

We see (Fig. 1) that Euclidean structures $\mathbb{R}^{2,0}$ on $\mathbb{R}^2$ induced by two Lorentzian equivalent frames $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{v_1^\alpha, v_2^\alpha\}$ are conformally inequivalent. In other words, we see that to any Lorentzian structure on $\mathbb{R}^2$ we can relate a one-parameter family of inequivalent conformal structures. [We recall that the set of all conformal structures on $\mathbb{R}^2$ can be given by elements of $GL_+(2, \mathbb{R})/GL(1, \mathbb{C}) \cong \Delta = \{z \in \mathbb{C};\ |z| < 1\}$.] This one-

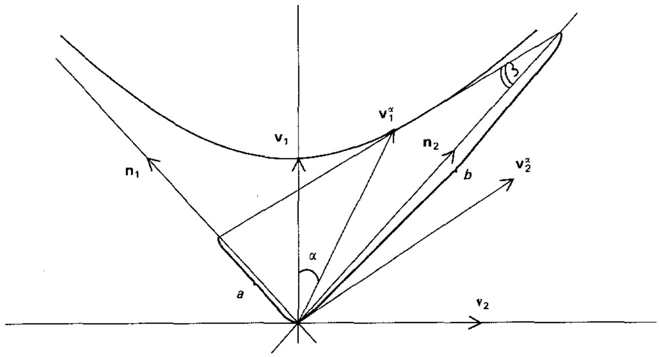


**Fig. 1**

parameter family corresponds to the set of the future-oriented timelike unit vectors of $\mathbb{R}^{1,1}$ and can also be parametrized by the parameter $\alpha$ of the $SO(1, 1)$ group.

Now let us notice that the light directions are orthogonal for all (mutually conformally inequivalent) Euclidean structures determined by the family $\{\mathbf{v}_1^\alpha, \mathbf{v}_2^\alpha\}_{\alpha \in (-\pi/4, \pi/4)}$. In other words, let us notice that we can introduce all of those Euclidean structures by frames $\{\mathbf{n}_1^\alpha, \mathbf{n}_2^\alpha\}$ which differ from each other merely by change of units in the same directions instead of by frames $\{\mathbf{v}_1^\alpha, \mathbf{v}_2^\alpha\}$. Namely, if $\beta$ denotes the angle between $\mathbf{v}_1^\alpha$ and $\mathbf{n}_2$ (see Fig., 1), then the ratio of the "$\alpha$ units" in the $\mathbf{n}_1$ and $\mathbf{n}_2$ directions measured in $\alpha = 0$ coordinates is

$$\frac{a}{b} = \operatorname{tg} \beta; \qquad \beta = \frac{\pi}{4} - \alpha \in \left(0, \frac{\pi}{2}\right) \tag{2.2}$$

Let us come back to our world sheet $\Sigma^L$ and to the immersion $X$, (2.1). It can be seen that any fixed timelike vector $e_0 \in \mathbb{R}^{1,D}$ defines a timelike vector field $\mathbf{v}_1$ on $\Sigma^L$ almost everywhere (or equivalently it determines a sections $s_0$ of the principal bundle $\xi_s$ of Lorentzian frames over $\Sigma^L$ a.e.). In this way we obtain a Riemannian structure, say $\Sigma_0$, on the underlying manifold $\Sigma$ of $\Sigma^L$. (This Riemannian structure has to be singular and we will discuss this point at the end of this section.) However, as in the $\mathbb{R}^2$ case, we obtain simultaneously a one-parameter family of conformally inequivalent Riemannian structures $\Sigma_\alpha$ on $\Sigma$ parametrized by $\alpha \in SO(1, 1)$ ($\Sigma_\alpha$ is determined by a section $s_\alpha = s_0 \circ \alpha$, $\alpha \in SO(1, 1)$, exactly in the same way as $\Sigma_0$ is determined by $s_0$). This family forms a curve $\{\Sigma_\alpha)_{\alpha \in (-\pi/4, \pi/4)}$

in the appropriate Teichmüller space which we call a $P$-line and we will see that this $P$-line is an infinite geodesic in the Teichmüller metric of $\mathcal{T}_{p,n}$.

When we fix a pair $(S_0, id)$ as an origin of the Teichmüller space $\mathcal{T}_{p,n}$, then any element $\tau \in \mathcal{T}_{p,n}$ can be considered as some concrete complex structure, say $S_1$, together with a homotopy class of quasiconformal maps from $S_0$ to $S_1$. Hence, by definition, all homeomorphisms between two points in $\mathcal{T}_{p,n}$ are homotopic quasiconformal maps which are differentiable almost everywhere. We recall that for a quasiconformal map $f\colon S_0 \to S_1$ the deviation from conformality at any differentiable point $x \in S_0$ [measured by the ratio $K_f(x)$ of the axes of the infinitesimal ellipse at $f(x)$, which is the image of an infinitesimal circle centered at $x$] is finite. For any quasiconformal map $f\colon S_0 \to S_1$ we define the global dilatation $K_f$ as the essential supremum of $K_f(x)$ over all $x \in S_0$. Teichmüller has shown (Ahlfors and Sario, 1960) that is any homotopy class of quasiconformal maps there exists a map with minimal global dilatation. It is called the Teichmüller map and its Beltrami differential $\mu$ (called in this case a Teichmüller differential) has the form

$$\frac{f_{\bar{z}}}{f_z} = \mu = k\frac{\phi}{|\phi|}, \qquad 0 \leqslant k < 1 \tag{2.3}$$

where $q = \phi\, dz^2$ is some holomorphic quadratic differential on $S_0$ which can have only zeros and poles of the first order in the punctures (here $f_z$ denotes $\partial f/\partial z$ and $f_{\bar{z}} \equiv \partial f/\partial \bar{z}$).

Any holomorphic quadratic differential $q$ on $S_0$ defines the horizontal line field [by the condition $\phi(z)\, dz^2 > 0$, for $\phi(z) \neq 0$] and the vertical line field [by $\phi(z)\, dz^2 < 0$]. Moreover, it defines a singular flat metric on $S_0$ induced by $|\phi(z)^{1/2}\, dz|$. Away from the singularities of $q$ (i.e., zeros and poles in punctures) we can locally introduce the so-called natural parameter

$$w = \int \sqrt{\phi}\, dz = u + iv \tag{2.4}$$

with respect to which the horizontal and vertical line fields are described by $v = \text{const}$ and $u = \text{const}$, respectively. In these coordinates the quadratic differential $q$ has the form

$$q = \phi(z)\, dz^2 = dw^2 \tag{2.5}$$

So, in the natural $q$ parameter $w$ our differential $q$ is represented by the function $\phi \equiv 1$. The Teichmüller map $f\colon S_0 \to S_1$ can be seen as a stretch map $(k, q)$ with respect to the natural parameter $w$ of $q$, i.e.,

$$\begin{aligned} u &\to u' = K^{-1/2} u \\ v &\to v' = K^{1/2} v \end{aligned} \qquad K = \frac{1+k}{1-k} \tag{2.6}$$

Now, the one-parameter family of Riemann surfaces to which $S_0$ is mapped by the $(k, q)$ stretch map (2.6) forms exactly an infinite geodesic $l(t)$ in the Teichmüller metric of $\mathcal{T}_{p,n}$, where $\tanh t = k$ (Nag, 1988). The geodesic $l(t)$ is usually called the Teichmüller line and the part $r(t)$ of $l(t)$ with $t \geqslant 0$ (i.e., $k \geqslant 0$) is called a Teichmüller ray.

Coming back to our $P$-line $\{\Sigma_\alpha\}_{\alpha \in (-\pi\mu/4, \pi/4)}$, we observe the following situation. For each $\alpha \in (-\pi/4, \pi/4)$ we have determined the fields $(\mathbf{n}_1^\alpha, \mathbf{n}_2^\alpha)$ of light vectors whose lengths vary with $\alpha$ according to (2.2). Let us introduce local coordinates $(x^\alpha, y^\alpha)$ related to vector fields $(\mathbf{n}_1^\alpha, \mathbf{n}_2^\alpha)$, respectively $(\mathbf{n}_1 \equiv \mathbf{n}_1^0, \mathbf{n}_2 \equiv \mathbf{n}_2^0)$. For each $\Sigma_\alpha$ its complex structure can be described by local coordinates

$$w^\alpha = x^\alpha + iy^\alpha \tag{2.7}$$

If we denote the appropriate local parameter $w_0$ on $\Sigma_0$ by $w_0 \equiv z = x + iy$, then we have

$$w^\alpha = w^\alpha(z) = x^\alpha + iy^\alpha = A(\operatorname{tg} \beta)^{1/2} x + i(\cot \beta)^{1/2}\, y) \tag{2.8}$$

where $\beta = \pi/4 - \alpha$, $\alpha \in (-\pi/4, \pi/4)$, and $A \in \mathbb{R}^*$. So we see that the mapping $w^0 \equiv z \to w^\alpha(z)$ is quasiconformal for each $\alpha \in (-\pi/4, \pi/4)$ and satisfies

$$w_{\bar{z}}^\alpha = \frac{\operatorname{tg} \beta - 1}{\operatorname{tg} \beta + 1}\, w_z^\alpha = \operatorname{tg} \alpha \cdot w_z^\alpha \tag{2.9}$$

(notice that $|\operatorname{tg} \alpha| < 1$). In other words, we realize that the parameter $z \equiv w^0$ (introduced by $\mathbf{n}_1, \mathbf{n}_2$) is the natural parameter of some quadratic differential $q$ on $\Sigma_0$, i.e.,

$$q = dz^2 \tag{2.10}$$

and that each $\Sigma_\alpha$ can be obtained from $\Sigma_0$ be a generalized affine stretch map $(k, q)$ with $k = \operatorname{tg} \alpha$, $\alpha \in (-\pi/4, \pi/4)$. In these coordinates the Beltrami differential $\mu = w_{\bar{z}}^\alpha / w_{\bar{z}}^\alpha$ has the form $\mu = \operatorname{tg} \alpha \cdot 1/1$, i.e., it is a Teichmüller differential. Thus, our one-parameter family $\{\Sigma_\alpha\}$ of Riemann surfaces determined by some concrete "observer" (i.e., fixed timelike unit vector $e_0 \in \mathbb{R}^{1,D}$) forms an infinite geodesic in the Teichmüller metric on $\mathcal{T}_{p,n}$. So we have shown that the following proposition is true.

*Proposition 2.1.* Any $P$-line is a Teichmüller line.

We can also see this using the approach of measured foliations. Namely, we have the following situation (from now on, for simplicity and without loss of generality we will assume that the manifold $\Sigma$ is compact).

The timelike vector field $\mathbf{v}$ on $\Sigma^L$ introduced above determiners a pair $\mathbf{n}_1, \mathbf{n}_2$ on lightlike vector fields. Now the pair of vector fields $\{\mathbf{n}_1, \mathbf{n}_2\}$ can be used not only to define a complex structure $J$ on $\Sigma$ [by $J \cdot \mathbf{n}_1(m) = \mathbf{n}_2(m)$], $m \in \Sigma$, but also to define a locally flat (singular) Riemannian metric $g$ on $\Sigma$, and a pair of measured transverse foliations. By Hubbard and Masur (1979), any pair of transverse measured foliations determines both a conformal structure $\Sigma_0$ on $\Sigma$ and some concrete holomorphic quadratic differential $q$ on $\Sigma_0$. Since any holomorphic quadratic differential defines a unique infinite Teichmüller geodesic, we can easily check that this geodesic is exactly our $P$-line $\{\Sigma_\alpha\}_{\alpha \in SO(1,1)}$.

The horizontal and vertical distributions of any holomorphic quadratic differential $q = \phi(z)\, dz^2$ are determined by a pair $\{\phi_1, \phi_2\}$ of local 1-forms $\phi_1 = \operatorname{Re} \phi^{1/2}\, dz$, $\phi_2 = \operatorname{Im} \phi^{1/2}\, dz$ which satisfy

$$\phi_i = \pm \phi_i', \qquad i = 1, 2 \tag{2.11}$$

on the overlap $U \cap U'$ of any two charts $U$, $U'$ on $\Sigma_0$. If the cocycle defined by (2.11) determines a trivial line bundle over $\Sigma_0$, then the differential $q$ is called orientable; if the corresponding bundle is not trivial, then $q$ is non-orientable. In the former case $q$ is the square of some holomorphic 1-form, i.e., $q = \omega^2$ and $\omega = \phi_1 + i\phi_2$. In this case the holonomy group of a metric (of zero curvature) which arises from $q$ is trivial insofar as we can construct (singular) global vector fields $\mathbf{n}_1, \mathbf{n}_2$ dual to $\phi_1, \phi_2$, respectively.

Since the Euler class of the underlying manifold $\Sigma$ does not vanish $(g \geqslant 2)$, we cannot construct a tangent line bundle over $\Sigma$. This means that the Lorentzian structure $\Sigma^L$ has to be a singular one or equivalently that any timelike vector field $\mathbf{v}$ on $\Sigma^L$ has to be singular. This implies that the two lightlike vector fields $\mathbf{n}_1(m)$ and $\mathbf{n}_2(m)$ determined by $\mathbf{v}(m)$ have a singularity at the same points of $\Sigma$ as $\mathbf{v}(m)$ as well as that these singularities are of the same kind. For any vector on $\Sigma_0$ the Poincaré–Hopf theorem tells us that the sum of its indices at zeros is equal to the homological Euler characteristic $\chi(\Sigma) = 2 - 2g$. On the other hand, by the Riemann–Roth theorem we know that the degree of the divisor of the distributions $\phi_i$, $i = 1, 2$, of holomorphic one-forms is equal to $2g - 2$, i.e., the singularities of vector fields $\mathbf{n}_i$, $i = 1, 2$, are at the same points as the zeros of holomorphic one-forms $\phi_i$, $i = 1, 2$, and they have the same degree.

## 3. HARMONIC $P$-LINES

Since in our approach to string theory we are giving primary relevance to a Lorentz structure on $\Sigma$, we should know if this fact implies some additional restrictions on the possible Riemannian structures on our world sheet $\Sigma$. According to the Nash–Green theorem (Green, 1970), there always exists an isometric embedding of $\Sigma^L$ into $\mathbb{R}^{k,k}$ with $k=50$. For physical reasons we assume that our world sheet can be isometrically immersed into a vector space of signature $(1, D)$, i.e., into at most 51-dimensional real vector space. Now let us notice that 51 is exactly the dimension of the Euclidean vector space $\mathbb{R}^{51}$ which admits an isometric embedding of any two-dimensional Riemannian manifold. So we see that the fact that we give fundamental relevance to the Lorentz structure does not introduce any additional restrictions onto possible Riemannian structures of our world sheet.

Let us consider the situation when the Riemannian structure on our world sheet related to a concrete "observer" $e_0 \in \mathbb{R}^{1,D}$ possesses some concrete properties. The most regular situation would be when $X$: $\Sigma \to \mathbb{R}^{1+D}$ realizes a minimal immersion into $\mathbb{R}^{1+D}$ [we use the same letter $X$ as in Section 2 to denote the isometric immersion of $\Sigma$ into $\mathbb{R}^{1+D}$ uniquely determined by (2.1) and by $e_0 \in \mathbb{R}^{1,D}$]. However, it is known that although any noncompact Riemannian 2-manifold admits a proper embedding into $\mathbb{R}^k$, $k>5$, by a harmonic map, it is not necessarily a conformal one. Moreover, there are no compact minimal submanifolds in $\mathbb{R}^n$, $n>3$. So we see that minimal immersion into $\mathbb{R}^{1+D}$ is not the case with high probability. The next, also very regular situation appears when an immersion $X$ into $\mathbb{R}^{1+D}$ realizes a minimal immersion into the hypersphere $S^D$ of $\mathbb{R}^{1+D}$. In this case the Gauss map associated to $X$ is a harmonic and homothetic one.

### 3.1. Harmonic Maps

Let $\Sigma$ be a surface equipped with a metric $h$ and let $M$ be an $n$-dimensional Riemannian manifold with metric $g$, $n>3$. A map

$$f: \ \Sigma \to M \tag{3.1}$$

is called conformal if the angle measurement associated with the induced metric $\tilde{g}$ on $\Sigma$ coincides with the angle measured with respect to $h$, whenever $\tilde{g}$ is nondegenerate. A map $f$ is minimal if it is conformal and extremal with respect to the ordinary area integral

$$S = \int_\Sigma d\delta_{\tilde{g}} \tag{3.2}$$

The differential of $f$ can be viewed as the $f^{-1}(TM)$ valued 1-form on $\Sigma$, i.e.,

$$df \in \Gamma(T^*\Sigma \otimes f^{-1}(TM)) \tag{3.3}$$

where $\Gamma(\cdot)$ denotes the set of sections of corresponding bundle and $T^*\Sigma$ denotes the cotangent bundle over $\Sigma$. The second fundamental form $B$ is the covariant differential of $df$:

$$B = \nabla df \tag{3.4}$$

and a conformal immersion $f$ is minimal if and only if its mean curvature vector field $H$

$$H^{\tilde{g}} \stackrel{\mathrm{df}}{=} \mathrm{tr}_{\tilde{g}}\, B \tag{3.5}$$

vanishes (Eells and Sampson, 1964).

The energy $E(f)$ is defined by

$$E(f) = \tfrac{1}{2}\int_{\Sigma} |df(x)|^2 \tag{3.6}$$

where $|df(x)| =: 2e(f)(x)$ is the Hilber–Schmidt norm of the linear map $df(x)$: $T_x M \to T_{f(x)} M$..A map $f$ is harmonic if and only if it is an extremal of the energy integral $E$. If we define the energy 1 metric $\gamma$ on $\Sigma$ as

$$\gamma = e(f)h \tag{3.7}$$

(it is the only metric among all conformal ones on $\Sigma$ for which $f$: $(\Sigma, \gamma) \to (M, g)$ has energy function 1), then we can say that a map $f$, (3.1), is minimal if and only if it is harmonic with the energy 1 metric equal to the induced metric i.e., $\gamma = \tilde{g}$. For harmonicity of $f$ it is enough when $\tilde{g}$ is a Codazzi tensor for the Riemannian metric $\gamma$ and the $\gamma$ mean curvature vector field

$$H^{\gamma} = \mathrm{tr}_{\gamma}\, B \tag{3.8}$$

vanishes.

In the local coordinates the harmonicity condition can be written as the Euler–Lagrange equation for the energy functional

$$\Delta f^{\mu} + {}^{M}\Gamma^{\mu}_{\gamma\rho} \frac{\partial f^{\rho}}{\partial \delta^{i}} \frac{\partial f^{\gamma}}{\partial \delta^{j}} h^{ij} = 0, \qquad \mu = 1, \ldots, n, \quad i, j = 1, 2 \tag{3.9}$$

They are nonlinear partial differential equations of elliptic type. Usually we introduce the so-called tension field $\tau(f) \in \Gamma(f^{-1}(TM))$ given by

$$\tau^{\mu}(f) = h^{ij}(\nabla df)^{\mu}_{ij}, \qquad \mu = 1, \ldots, n, \quad i, j = 1, 2 \tag{3.10}$$

and we write equation (3.9) in the form

$$\tau^{\mu}(f)=0 \tag{3.11}$$

If $f$ is an isometric immersion of $\Sigma$ into $\mathbb{R}^n$ and if $g$ denotes its corresponding Gauss map

$$g\colon\ \Sigma \to G_{2,n}$$

(where $G_{2,n}$ is the Grassmann manifold of all oriented planes through a point 0 in an $n$-dimensional Euclidean space $\mathbb{R}^n$), then the tension field of $g$ is equal to (Ruth and Vilms, 1970; Takahashi, 1966; Muto, 1980)

$$\tau(g)=\nabla H$$

So we see that if $\Sigma$ is immersed with parallel mean curvature vector field, then the Gauss map $g$ is a harmonic one. If $f$ realizes a minimal immersion into the hypersphere $S^{n-1}\subset\mathbb{R}^n$, then its Gauss map $g$ has to be additionally homothetic. In this case $f$ satisfies

$$\Delta f=\lambda f$$

with $\lambda=2/a^2$ and $a$ is a radius of $S^{n-1}$. The components of the mean curvature vector are eigenfunctions of the Laplacian on $(\Sigma, h)$ belonging to the same (after a suitable parallel displacement) eigenvalue $\lambda$. Since each surface $(\Sigma, h)$ is an Einstein manifold, i.e., its Ricci tensor $K_{ij}$ satisfies $K_{ij}=\varkappa h_{ij}$ and since the Gauss map is homothetic, i.e., $G_{ij}=c^2 h_{ij}$ [where $G_{ij}$ is a metric on $g(\Sigma)$]

$$\lambda=\tfrac{1}{2}(c^2+\varkappa)$$

So in this case the Laplacian on $(\Sigma, h)$ has to have at least one eigenvalue $\lambda>\varkappa/2$ of multiplicity $\geqslant 3$. In particular, if $(\Sigma, h)$ is such that the immersion is full in $\mathbb{R}^n$, then there exists at least one eigenvalue $\lambda>\varkappa/2$ of multiplicity $\geqslant n$.

### 3.2. Harmonic *P*-Lines

Let us come back to an immersion of our world sheet $\Sigma^L$ into $\mathbb{R}^{1,D}$. The Minkowski structure of $\mathbb{R}^{1,D}$ together with a fixed timelike vector $e_0\in\mathbb{RT}^{1,D}$ determine a unique Euclidean structure $\eta$ on $\mathbb{R}^{1+D}$. Of course $\eta=\operatorname{diag}(1, 1, \ldots, 1)$ in any of its orthogonal bases. Let $\varepsilon$ denote one of the orthonormal bases whose first component is equal to $e_0$.

The metric $I$ introduced on $\Sigma$ by $X$ can be written

$$I_{ij} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \delta^i} \frac{\partial X^\nu}{\partial \delta^j}$$

Let us represent $\Sigma_0$ by $(\Sigma, h^0)$ and $\Sigma_\alpha$ by $(\Sigma, h^\alpha)$, respectively. We recall that $h^0$ and $h^\alpha$ are locally flat, singular Riemannian structures on $\Sigma$ uniquely determined by the quadratic differential $q$ (namely $\eta^0$ is $q$-metric and $h^\alpha$ is $q_\alpha$-metric, where $q_\alpha$ is the terminal differential of the appropriate Teichmüller map given by $q$). The vectors $\partial X/\partial\delta_0$ and $\partial X/\partial\delta_1$ are orthogonal in $(\mathbb{R}^N, \eta)$, $N = D + 1$, and have the same length [i.e., $I_{ij} = \text{diag}(1, 1)$ in these parameters]. Let $X$: $(\Sigma, h^0) \to (\mathbb{R}^N, \eta)$ be the isometric immersion which realizes a minimal immersion into a sphere $S^{N-1} \subset (\mathbb{R}^N, \eta)$. From the previous section we know that this means that

$$\frac{\partial^2 X^\mu}{\partial \delta_0^2} + \frac{\partial^2 X^\mu}{\partial \delta_1^2} = -\lambda X^\mu, \qquad \lambda > 0 \tag{3.12}$$

and the associated Gauss map is harmonic and homothetic.

Let us pass to a complex structure on $\Sigma$ related to $\Sigma_\alpha = (\Sigma, h^\alpha)$. The natural $q_\alpha$ coordinates are $\{\delta'_0, \delta'_1\} = \{\delta_0, \text{tg}\, \alpha\delta_1\}$ and we have

$$h^0_{ij} = \text{diag}(1, 1) = I_{ij}, \qquad h^{0\prime}_{ij} = \text{diag}(1, \text{ctg}^2\, \alpha) = I'_{ij} \tag{3.13}$$

$$h^\alpha = \text{diag}(1, \text{tg}^2\, \alpha) \qquad h^{\alpha'} = \text{diag}(1, 1) \tag{3.13'}$$

$$\alpha \in (0, \pi/2)$$

here the index 0 corresponds to $\alpha = \pi/4$ and the prime denotes appropriate entries with respect to the $\{\delta'_0, \delta'_1\}$ parameters. Now if we consider $X$: $\Sigma \to \mathbb{R}^N$ as

$$X\colon \quad (\Sigma, h^0) \to (\mathbb{R}^N, \eta) \tag{3.14}$$

then the energy $E(X) = 1$, $e(X) = 1$ and the energy 1 metric $\gamma = h_0 = I$. If we treat pointwise the same map $X$ as

$$X\colon \quad (\Sigma, h^\alpha) \to (\mathbb{R}^N, \eta) \tag{3.15}$$

then

$$E(X) = \frac{1}{2} \int_\Sigma h^{\alpha' ij} \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \delta'^i} \frac{\partial X^\nu}{\partial \delta'^j} (\det h^{\alpha'})^{1/2}\, d\delta'_0 \wedge d\delta'_1 \neq 1 \tag{3.16}$$

and

$$e(X)=\frac{1}{2}h^{\alpha' ij}\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\delta'^i}\frac{\partial X^\nu}{\partial\delta'^j}=\frac{1}{2}(1+\cot^2\alpha) \tag{3.17}$$

So the energy 1 metric $\gamma'_\alpha$ of the immersion (3.15) in prime coordinates is

$$\gamma^{\alpha'}=\tfrac{1}{2}(1+\cot^2\alpha)h^{\alpha'} \tag{3.18}$$

and the Laplace–Beltrami operator associated to $\gamma^{\alpha'}$ is

$$\Delta_{\gamma\alpha'}= -\frac{1}{2}(1+\cot^2\alpha)\left(\frac{\partial^2}{\partial\delta_0'^2}+\frac{\partial^2}{\partial\delta_1'^2}\right) \tag{3.19}$$

*Proposition 3.1.* If map $X$: $(\Sigma, h^0)\to(\mathbb{R}^N,\eta)$ realizes the minimal immersion of $\Sigma_0$ into a hypersphere $S^{N-1}_{r0}\to(\mathbb{R}^N,\eta)$ then $X$: $\Sigma^L\to\mathbb{R}^{1,N-1}$ satisfies the linear wave equation $\partial^2X/\partial\delta_0^2-\partial^2X/\partial\delta_1^2=0$ if and only if $X$: $(\Sigma, h^{\alpha'})\to S^{N-1}_r$ is harmonic for each $\alpha\in(0,\pi/2)$.

*Proof.* It is based on the following Milnor theorem (1983).

*Theorem 3.1.* The immersion $X$: $(\Sigma, h^{\alpha'})\to(\mathbb{R}^N,\eta)$ satisfies $\mathrm{Cod}(\gamma^{\alpha'}, I')$ and $\Delta_{\gamma^{\alpha'}}X=\lambda_\alpha X$ for $\lambda_\alpha>0$ if and only if $X$: $(\Sigma, h^{\alpha'})\to S^{N-1}\subset(\mathbb{R}^N,\eta)$ is harmonic with $r^2=2/\lambda_\alpha$.

The notation $\mathrm{Cod}(\gamma^{\alpha'}, I')$ means that $I'$ satisfies the classical Codazzi–Minardi equations with respect to $\gamma^{\alpha'}$ (as a metric). This forces the tangent component of the tension field of the immersion (3.15) to vanish. When $X$ is conformal, then $\gamma^{\alpha'}=I'$, so that $\mathrm{Cod}(I', I')$ is automatic. However, we meet this case only for $\alpha=\pi/4$, i.e., for $h^0$.

In a general case, with any immersion of $(\Sigma, h^{\alpha'})$ into $(\mathbb{R}^N,\eta)$ there is an associated quadratic differential $\tilde{q}=\varphi\, dz'^2$, $z'=\delta_1'+i\delta_0'$ on $(\Sigma, h^{\alpha'})$. Namely, if the induced metric of an immersion is

$$I'=E\, d\delta_1'^2+2F\, d\delta_1'\, d\delta_2'+G\, d\delta_0'^2 \tag{3.20}$$

then

$$\varphi=E-G-2iF \tag{3.21}$$

(note that $\tilde{q}=0$ if the immersion is a conformal one). In our case for the map (3.15) we have

$$\tilde{q}=(1-\cot^2\alpha)\, dz'^2 \tag{3.22}$$

i.e., $\tilde{q}$ is a holomorphic quadratic differential on $\Sigma_\alpha \equiv (\Sigma, h^\alpha)$. Since it is known that $\tilde{q}$ is holomorphic if and only if $\mathrm{Cod}(\gamma^{\alpha'}, I')$, we see that the first property of the Milnor theorem is fulfilled for any $\alpha \in (0, \pi/2)$. In the language of Teichmüller space, we can say that Riemann surface $\Sigma_\alpha$ can be obtained from $\Sigma_0$ by the Teichmüller map determined by the unique holomorphic quadratic differential on $\Sigma_0$ for which $(\delta_0, \delta_1)$ are the natural parameters. Now it is easy to see that the just obtained holomorphic quadratic differential $\tilde{q}$, (3.22), is nothing else but (up to a positive constant) the terminal differential on $\Sigma_\alpha$ of this Teichmüller map.

Now let $X$: $(\Sigma, h^0) \to (\mathbb{R}^N, \eta)$ satisfy

$$\frac{\partial^2 X}{\partial \delta_0^2} - \frac{\partial^2 X}{\partial \delta_1^2} = 0 \tag{3.23}$$

This fact together with formula (3.12) and

$$\frac{\partial^2}{\partial \delta_0'^2} + \frac{\partial^2}{\partial \delta_1'^2} = \frac{\partial^2}{\partial \delta_0^2} + \cot^2 \alpha \frac{\partial^2}{\partial \delta_1^2} \tag{3.24}$$

give

$$\frac{\partial^2 X}{\partial \delta_0'^2} + \frac{\partial^2 X}{\partial \delta_1'^2} = \frac{\lambda}{2} (1 + \cot^2 \alpha) X \tag{3.25}$$

so according to Theorem 3.1 [we recall that we have $\mathrm{Cod}(\gamma^{\alpha'}, I')$], the immersion $X$: $(\Sigma, h^{\alpha'}) \to S^{N-1}$ is a harmonic one. Conversely, if $X$ satisfies (3.12) and (3.25) (i.e., if the corresponding map into hypersphere is harmonic), then

$$\frac{1}{2} (\cot \alpha - 1) \frac{\partial^2 X}{\partial \delta_0^2} + \frac{1}{2} (1 - \cot^2 \alpha) \frac{\partial^2 X}{\partial \delta_1^2} = 0 \tag{3.26}$$

i.e., $X$: $(\Sigma, h^0) \to (\mathbb{R}^N, \eta) \equiv X\colon \Sigma^L \to \mathbb{R}^{1,n}$ sxatisfies $\partial^2 X/\partial \delta_0^2 - \partial^2 X/\partial \delta_1^2 = 0$. So our assertion is proven.

If for each $\alpha \in (0, \pi/2)$ there exists an affine map $A_\alpha \in GL^+(N, \mathbb{R})$ such that

$$X\colon \quad (\Sigma, h^{\alpha'}) \to (\mathbb{R}^N, \eta^\alpha) \tag{3.27}$$

is a conformal one [here $\eta^\alpha = \mathrm{diag}(1, 1, \ldots, 1)$ in a base $\varepsilon' = \varepsilon \cdot A_\alpha$], then the word "harmonic" in our proposition can be changed for "minimal." We meet such a situation if, for example, the quadratic differential on $\Sigma_0$ determined by a pair of measured foliations $(\delta_0, \delta_1)$ is a Jenkins differential with only one horizontal and only one vertical cylinder.

The case when the $P$-line consists of elements which, with respect to their "natural parameters," can be harmonically immersed into the corresponding hypersphere seemes to be appealing for its regularity and simplicity. However, in this case we have to find an other than Nambu–Goto action for the world sheet of our string. Namely, let us notice that although equation (3.23) has the same form as the linear wave equation which minimalizes the Nambu–Goto action, it is written in "light coordinates" instead of "time and space coordinates." It is easy to check that if assumptions of Proposition 3.1 are fulfilled (i.e., our $P$-line is harmonic), then Nambu–Goto equations cannot be satisfied at all. So if we believe that the Nambu–Goto action describes a free string, then the harmonic $P$-line could be related to a self-interacting string.

## 4. JENKINS–STREBEL DIFFERENTIALS AND DECAY OF A WORLD SHEET

Let $\{\delta_0, \delta_1\}$ be local coordinates on our world sheet $\Sigma^L$ which are determined by the light vector fields $\mathbf{n}_1$ and $\mathbf{n}_2$ as before. As we have seen in Section 2, they form leaves of a pair of transverse measured foliations. These leaves are horizontal and vertical trajectories of some concrete holomorphic quadratic differential $q = dz^2$, where $z = i\delta_0 + \delta_1$. Local natural parameter on the Riemann surface $\Sigma_k$ belonging to the Teichmüller $P$-line $l_q$ are

$$\delta_0' = (\cot \alpha)^{1/2} \delta_0, \qquad \alpha \in (0, \pi/2)$$

$$\delta_1' = (\operatorname{tg} \alpha)^{1/2} \delta_1, \qquad z' = i\delta_0' + \delta_1'$$

If our $P$-line is a harmonic one, then we have (almost everywhere)

$$\frac{\partial^2 X^\mu}{\partial \delta_0^2} = \frac{\lambda}{2} X^\mu, \qquad \frac{\partial^2 X^\mu}{\partial \delta_1^2} = \frac{\lambda}{2} X^\mu, \qquad \mu = 1, \ldots, 1 + D$$

i.e., $X^\mu$ are periodic functions of $\delta_0$ and $\delta_1$ (a.e.). We see that both their periods $c_i$, $i = 0, 1$, are equal, i.e.,

$$c_0 = c_1 = c = \frac{2.2^{1/2}\pi}{(-\lambda)^{1/2}}$$

But this means that our "physical" differential $q$ has to have closed horizontal and closed vertical trajectories. Holomorphic quadratic differentials which satisfy this property are called Jenkins–Strebel differentials.

### 4.1. Jenkins–Strebel Differentials

Let us recall some properties of a quadratic differential $q$ with closed horizontal trajectories (Strebel, 1984) (vertical trajectories of $q$ are the horizontal ones of $-q$). For such quadratic differentials the critical graph $\Gamma_q$, i.e., the set of critical trajectories with their critical endpoints (zeros and simple poles in punctures) is compact. This implies that $\Sigma_0 - \Gamma_q$ is covered by ring domains $R_i \subset \Sigma_0$. A ring domain $R_i$ in $\Sigma_0$ is said to be a homotopy type $\gamma$ if a Jordan curve $\gamma_0 \subset R_i$ which separates its two boundary components is freely homotopic to $\gamma$.

Let $\xi$ be any local parameter on $\Sigma_0$. In this parameter our quadratic differential $q$ will have the form

$$q = dz^2 = \phi(\xi)\, d\xi^2 \tag{4.1}$$

where $z$ is its natural parameter. When we cut the characteristic ring domain $R_i$ of $q$ along a vertical arc connecting its two boundary components, then a branch of $\int \sqrt{\phi}\, d\xi = \Phi(\xi)$ maps it into a horizontal rectangle $S_i$ in the $z = \delta_i + i\delta_0$ plane (Fig. 2).

Since the horizontal and vertical trajectories of $q$ are smooth curves along which

$$\arg dz^2 = \arg \phi(\xi)\, d\xi^2 = \begin{cases} 0 & \text{for horizontal arc} \\ \pi & \text{for vertical arc} \end{cases}$$

the horizontal and vertical sides of $S_i$ have lengths

$$a_i = \int_\gamma |\phi(\xi)|^{1/2}\, |d\xi| \tag{4.2}$$

and

$$b_i = a_i M_i \tag{4.3}$$

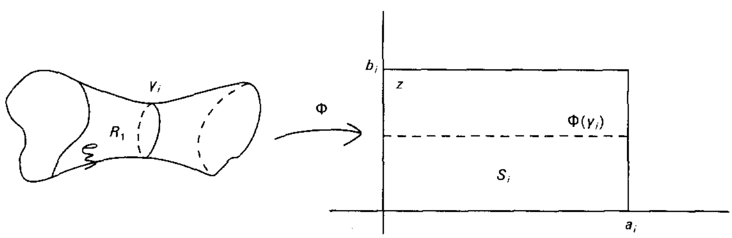


**Fig. 2**

[here $M_i$ is the modulus of a ring domain $R_i$; it is defined as $M_i = (1/2\pi)\log(r_2/r_1)$ if $r_1 < |\xi| < r_2$ is its conformally equivalent annulus]. To demonstrate some properties of Jenkins–Strebel differentials we will, for simplicity, assume that $q$ has only one horizontal cylinder. The norm of $q = dz^2 = \varphi(\xi)\, d\xi^2$ which has only one ring domain $R$ is the Euclidean area of $S$, i.e.,

$$\|q\| = \int_R |\varphi(\xi)|\, du\, dv = \int_S d\delta_0\, d\delta_1 = ab \tag{4.4}$$

$R$ is swept out by the closed horizontal trajectories of $q$ which have length (in the $q$-metric $|\varphi|^{1/2}\, |d\xi|$) equal to $a$, and $b$ is the height of $R$. Since the modulus $M$ of $R$ is a conformal invariant (and is uniquely determined by a simple loop $\gamma$ on $\Sigma_0$), the height of the cylinder of a given homotopy type on $\Sigma_0$ uniquely determines a quadratic differential $q$.

Let us consider a holomorphic map

$$p(\xi) = \exp\left\{-\frac{2\pi i}{a}\int [\varphi(\xi)]^{1/2}\, d\xi\right\} = W \tag{4.5}$$

which transforms the characteristic ring $R$ into the maximal $q$-annulus $A$ of type $\gamma$. In terms of the parameter $W$, $q$ has the representation

$$q = dz^2 = \varphi(\xi)\, d\xi^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{1}{W^2}\, dW^2 \tag{4.6}$$

and the affine stretch of magnitude $K^{1/2} = (\mathrm{tg}\,\beta)^{1/2}$, $\beta \in [\pi/4, \pi/2)$, along the horizontal trajectories of $q$ is realized in the annulus $A$ as a contraction along concentric circles (Marden, 1980) (Fig. 3).

The Teichmüller map $f$ (with Beltrami coefficient $\mu = k\,\bar{\varphi}/|\varphi|$) from $\Sigma_0$ to another Riemann surface on a Teichmüller line $l_q$ can be expressed as

$$f = \psi \circ \mathscr{A} \circ p \tag{4.7}$$

where $p$ is the map (4.5) of the cut $\Sigma_0$ onto the annulus $A$, and $\mathscr{A}$ is its radial contraction,

$$\mathscr{A}(W) = W_k = W|W|^{1/k-1} \tag{4.8}$$

and $\psi$ is the image annulus into the Riemann surface that is obtained by identifying appropriate arcs on its boundary.

If we normalize $p$ such that $p(\Sigma_0 - \Gamma_q) = \{r < |W| < 1\}$, then the situation is as shown in Fig. 4.

To understand better what is happening when $K \to \infty$, let us consider

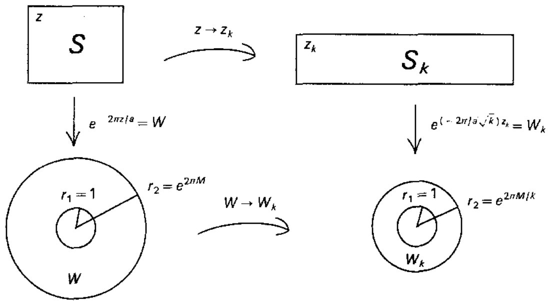


**Fig. 3**

following maps (Masur, 1975). First let us normalize the map $p$: $R \to A$ such that $A = \{r < |W| < 1\}$. Now let us cut $A$ along the Jordan curve $\alpha$ corresponding to $|W| = \sqrt{r}$ (i.e., along the central curve of $A$). Let $A^1$ and $A^2$ denote the annuli $r < |W| < \sqrt{r}$ and $\sqrt{r} < |W| < 1$, respectively. Let us reparametrize $A$ by $W' = r/W$. Now, for each $K > 1$ we glue the annulus $B^1_K = \{W' \mid (r^K)^{1/2} < |W'| < r^{1/2}\}$ to $A^1$ along $\alpha$ without a twist to form $A^1_K$ and glue $B^2_K = \{W \mid (r^K)^{1/2} < |W| < r^{1/2}\}$ to $A^2$ along $\alpha$ without a twist to form $A^2_K$. Finally, $W' = (r^K)^{1/2} e^{-i\theta}$ and $W = (r^K)^{1/2} e^{i\theta}$ are identified to form $R^K$, the so-called $K$ model of $R$.

An affine stretch $W' \to W' |W'|^{K-1}$ of $A^1$ onto $A^1_K$ and $W \to W |W|^{K-1}$ of $A^2$ onto $A^2_K$ extends to the critical trajectories of $q = dz^2 = \varphi(\xi)\, d\xi^2$ to become a quasiconformal map $f_K$ of $\Sigma_0$ onto $\Sigma_K$. It can be seen that the $K$-model $\Sigma_K$ of $\Sigma_0$ is conformally equivalent to $f_K(\Sigma_0)$, where $f_K$ is the Teichmüller map with complex dilatation $\mu = -k\bar{\varphi}/|\varphi|$,

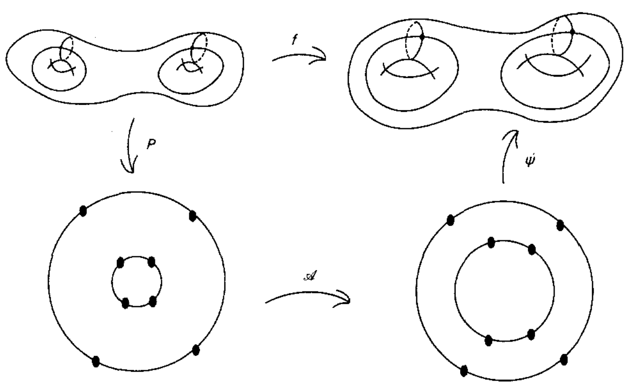


**Fig. 4**

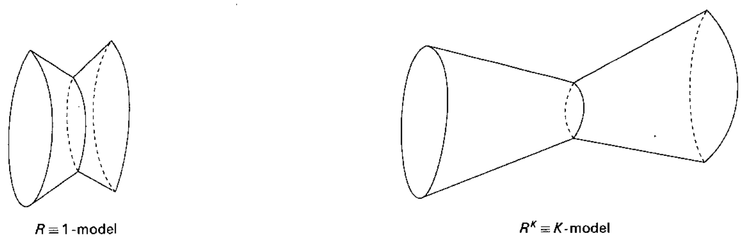


**Fig. 5**

$k \geqslant 0$, $k = (K-1)/(K+1)$. There is a conformal embedding $i_K$ of $\Sigma_0 - \alpha \to \Sigma_K$ which embeds each $A^i$ in $A^i_K$, $i = 1, 2$, by

$$i_K(W) = W, \qquad i_K(W') = W'$$

The two sides of $\alpha$ correspond under $i_K$ to the two curves on $\Sigma_K$ along which $B^i_K$ is glued to $A^i$. This procedure indicates that at the "end" of a geodesic ray we should glue the punctured discs $B^1 = \{W' \mid 0 < |W'| < \sqrt{r}\}$ and $B^2 = \{W \mid 0 < |W| < \sqrt{r}\}$ to the two sides of $\alpha$ without twist. This model serves as a model for the surfaces (or surface—depending on whether $\alpha$ is dividing or not) with additional 2-punctures on the boundary of Teichmüller space $\mathscr{T}_{p,n}$. However, to investigate the problem of convergence of geodesic rays $(K, -q)$ where $q$ is a nonzero quadratic differential with closed horizontal trajectories we should pass to the Bers embedding of $\mathscr{T}_{p,n}$ into the finite-dimensional complex Banach space $B_2(\Gamma, L)$ of bounded quadratic differentials.

### 4.2. Boundary of $T(\Sigma_0)$ and *b*-Groups

For any $\varphi \in B_2(\Gamma, L)$ the solution $W_\varphi$ of the Schwarzian differential equation $\{W_\varphi, z\} = \varphi$ $[\{W, z\} = (W''/W')^1 - \frac{1}{2}(W''/W')^2]$ induces a homomorphism $\theta_\varphi \colon \Gamma \to$ Möb

$$\theta_\varphi(\Gamma) = \Gamma^\varphi = W_\varphi \Gamma W_\varphi^{-1} \subset \text{Möb}$$

The group $\Gamma^\varphi$ is called the monodromy group of the differential equation $\{W_\varphi, z\} = \varphi$. For $\varphi \in \mathscr{T}_p \cong T(\Gamma) \cong T(\Sigma_0)$ the group $\Gamma^\varphi$ is quasi-Fuchsian with precisely one unbounded component $\Delta_1 = W_\varphi(L)$ of its domain of discontinuity $\Omega$ and one bounded component $\Delta_2 = W_\varphi(U)$. Now $\Delta_1/\Gamma^\varphi \cong \bar{\Sigma}_0$, whereas $\Delta_2/\Gamma^\varphi$ is conformally equivalent to the "variable" Riemann surface $\Sigma_\varphi \equiv \varphi \in \Phi(T(\Gamma))$ (Gardiner, 1987).

The image of the Bers map $\Phi$ is bounded in $B_2(\Gamma, L)$ and the identification of $\mathscr{T}_p$ with $\Phi(\mathscr{T}_p) B_2(\Gamma, L)$ determines a boundary $\partial\mathscr{T}_p$ which is called the complex boundary and which depends on the choice of the origin $\Sigma_0$ (or equivalently on the choice of $\Gamma \subset \text{Möb}_{\mathbb{R}}$). For each $\phi \in \partial \mathcal{T}_p \to B_2(\Gamma, L)$ the group $\Gamma^\phi$ is always Kleinian and has only one invariant component $\Delta_1 = W_\phi(L)$ of $\Omega$. Such groups $\Gamma^\phi$ are called $b$-groups. Any other component $\Delta$ of $\Omega$ of $\Gamma^\phi$ is simply connected and not invariant. If $\Gamma^\phi_\Delta$ denotes the stabilizer of $\Delta$ in $\Gamma^\phi$, then $\Delta/\Gamma^\phi$ is a finite Riemann surface of type $(p', n')$. So, for $\phi \in \partial \mathcal{T}_p$ the component $\Delta_2 = W_\phi(U)$ is the (perhaps empty) union of all noninvariant components of discontinuity. In this case we write

$$\Omega/\Gamma^\phi = \Delta_1/\Gamma^\phi + \Delta'/\Gamma^\phi_{\Delta'} + \Delta''/\Gamma^\phi_{\Delta''} + \cdots + \Delta^k/\Gamma^\phi_\Delta$$

or

$$\Omega/\Gamma^\phi = \bar{\Sigma}_0 + S_1 + S_2 + \cdots + S_k$$

It turns out that "almost all" $b$-groups are totally degenerate, that is, satisfy $\Delta_1 = \Omega$. A regular boundary group $\Gamma^\phi$ represents a Riemann surface $\bar{\Sigma}_0$ and one or more surfaces $S_1, S_2, \ldots, S_k$ which may be thought to have been obtained by drawing allowable Jordan curves on $\Sigma_0$ and then contracting each to a point on $\Sigma_0$. So we can say that for a regular boundary point ($b$-group) $\Delta_2/\Gamma^\phi$ is a finite union of Riemann surface which topologically may be derived from $\Sigma_0$ by cutting along an admissible system of Jordan curves $\gamma_1, \ldots, \gamma_k$ and by gluing a punctured disc to each side of each cut (Abikoff, 1980). (All of these considerations can be generalized to any Teichmüller space $\mathcal{T}_{p,n}$, where $n$ is the number of punctures. This means that we will get exactly the same result for world sheets which can be related to Riemann surfaces with $n$ punctures.)

### 4.3. *P*-Lines and Jenkins–Strebel Differentials

Let $\Sigma^L$ be the world sheet of same string object and let $l_q$ be a Teichmüller $P$-line through $\Sigma_0$ determined by a concrete "observer." We know from Section 3 that if $\Sigma_0$ is minimally immersed into a hypersphere $S^D \to \mathbb{R}^{1+D}$ and if our $P$-line is a harmonic one, then the quadratic differential $q$ has to be a Jenkins–Strebel differential. In the future we will assume that each $P$-line is determined by Jenkins–Strebel differentials, but we will not necessarily require that this is a harmonic line. (The harmonicity is not a necessary condition to have a $P$-line related to a Jenkins–Strebel differential.)

Now, if $l_q$ is a Teichmüller $P$-line through $(\Sigma_0, \text{id})$, determined by a concrete observer $e_0 \in \mathbb{R}^{1,D}$, let $(K, -q)$ denote a Strebel ray through the point $(\Sigma_0, \mathrm{id})$ and let $f_K$: $\Sigma_0 \to \Sigma_K$, $K = K(\alpha)$, be the corresponding Teichmüller map. In the Bers embedding this ray is represented by points

$$\varphi_K = \{W_K, z\} \in B_2(\Gamma, L) \tag{4.9}$$

and for each $\varphi_K$ the group $\Gamma^{\varphi_K}$ is the group of simultaneous uniformizations of $\bar{\Sigma}_0$ and $\Sigma_K$ (here $W_K$ is a unique quasiconformal automorphism of $\hat{\mathbb{C}}$ which fixes 0, 1, and $\infty$ and has properties that $W_K|_{\mathscr{U}}$ has the same Beltrami differential as $f_K$ and $W|_L$ is holomorphic).

Masur (1975) has shown that the endpoint of the Strebel ray $(K, -q)$ is given by the punctured model $\tilde{\Sigma}_0$ or $\Sigma_0$ and that there exists a regular boundary point $\bar{\varphi} \in \partial\mathscr{T}_p \subset B_2(\Gamma, L)$ such that

$$\tilde{\Sigma}_0 \cong \Delta_2/\Gamma^{\bar{\varphi}}$$

i.e., $\tilde{\Sigma}_0$ denotes the corresponding union of appropriate Riemann surfaces described in Sections 4.1 and 4.2. For example, if the Jenkins–Strebel ray is characterized by more then one ring domain, its endpoint can consist of two or more Riemann surfaces (see Fig. 6).

From the physical point of view the existence of such $P$-lines, determined by a quadratic differential with closed trajectories (harmonic or not), seems to be the most plausible. In this case we have that any physical object which is related to a Lorentzian world sheet $\Sigma^L$ cannot be stable. It has to be created—what is described by the so-called opening procedure for the horizontal cylinder of a Jenkins–Strebel ray $(k, q)$—and it has to decay—what is described by the endpoint of JS ray $(k, -q)$. Since for any $P$-line $l_q$ we have the identification of $k = (K-1)/(K+1)$ with $K = \operatorname{tg}\beta$, where $\beta = \pi/4 - \alpha \in [\pi/4, \pi/2)$ and $\alpha$ has a well-defined physical interpretation (see Section 2), the time orientability of $\Sigma^L$ guarantees that the notions of "creation" and "decay" are definitely distinguished and well defined. The

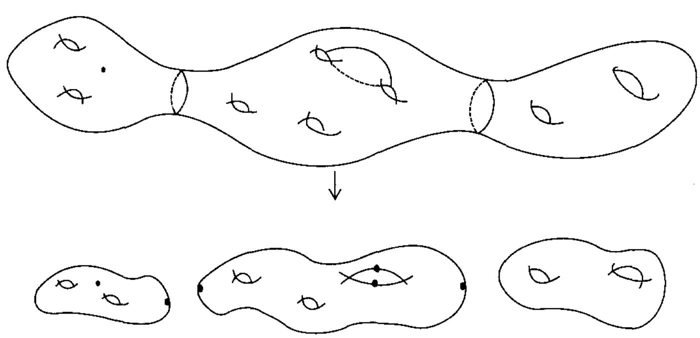

**Fig. 6**

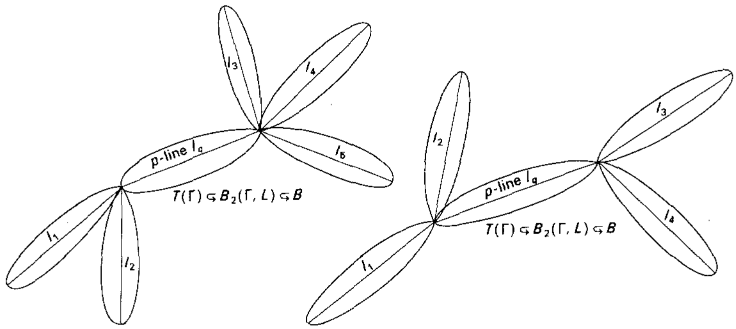


**Fig. 7**

endpoints of the ray $(k, q)$ can be interpreted as objects which take part in some collision process. Similarly, any element $S_1$ of $\tilde{\Sigma}_0$, i.e., any element of decay, can take part in some other collision process, i.e., be one of the elements of some other opening procedure (see example in Fig. 7).

## 5. DISCRETE SPACE-TIME

In this section we will show that if the Jenkins–Strebel differential determining the $P$-line has only one cylinder of each type and if their heights and circumferences are equal to each other (we meet such a situation, for example, for a harmonic $P$-line), then only discrete set $\mathcal{O} \subset H = \{e \in \mathbb{R}^{1,D}; e^2 = 1\}$ of "observers" can observe the same physical rules. From this fact we can conclude that space-time has to be discrete.

The quadratic differentials with closed horizontal and closed vertical trajectories each determining only one cylinder are dense (Masur, 1979). This fact suggests that our assumption about one cylinder of each type seems to be quite reasonable and we should investigate such a case in any event.

### 5.1. Absolutely Extremal Self-Mapping

Let $\Sigma_0$ be a Riemann surface of type $(p, n)$, $p > 1$, and let $\Gamma$ be the Fuchsian group such that $\Sigma_0 \cong U/\Gamma$, where $U$ is the upper half-plane of $\mathbb{C}$. Let us introduce the following notation:

$$Q = \{\text{the group of quasiconformal automorphisms of } U\}$$

$$Q_n = \{\omega \in Q; \omega(0) = 0, \omega(1) = 1, \omega(\infty) = \infty\}$$

$$Q(\Gamma) = \{\omega \in Q; \omega\Gamma\omega^{-1} \subset \text{Möb}_{\mathbb{R}}\}$$

$$Q_0 = \{\omega \in Q; \omega \cong \text{id}\}$$

We say that $\omega_1 \sim \omega_2$ if and only if

$$\omega|_{\hat{\mathbb{R}}} = \omega_2|_{\hat{\mathbb{R}}}$$

Now the set $\{[\omega]\}$ of equivalence classes of $\omega \in Q(\Gamma) \cap Q_n$ serves as a model $T(\Gamma)$ of the Teichmüller space $\mathscr{T}_{p,n}$. Each element $\omega \in Q$ induces a map

$$\omega_*: \quad Q_n \to Q_n; \qquad \omega_*(w) = a \cdot w \cdot \omega^{-1}; \qquad \forall w \in Q_n \tag{5.1}$$

where $a$ is some element of Möb depending on $w$, which transforms $w \cdot \omega^{-1}$ onto a normalized element $\omega_*(w) \in Q_n$. Any such map is holomorphic and it is an isometry on a complete metric space $Q_n$ with a metric induced by the Teichmüller distance. If $\omega \in Q_n$ has property that

$$\omega \Gamma \omega^{-1} \subset \Gamma$$

i.e., if $\omega$ belongs to the normalizer $N(\Gamma)$ of $\Gamma$ in $Q_n(\Gamma) = Q(\Gamma) \cap Q_n$, then the modular group $M(\Gamma)$ is defined as

$$M(\Gamma) = N(\Gamma)/Q_0(\Gamma) \tag{5.2}$$

where $Q_0(\Gamma) = N(\Gamma) \cap Q_0$ is the centralizer of $\Gamma$ in $N(\Gamma)$. The group Mod $\Gamma$ acts on $T(\Gamma)$ and if the signature $(p, n)$, $p > 1$, is not $(2, 0)$, then this action is effective.

Any $\omega \in N(\Gamma)$ induces a diffeomorphism of $\Sigma_0 = U/\Gamma$ and conversely, any $f \in \mathrm{Diff}_+ \Sigma_0$ lifts to some element in $N(\Gamma)$. The projection $\Pi$: $U \to U/\Gamma \cong \Sigma_0$ induces the map

$$\Pi_*: \quad N(\Gamma) \to \mathrm{Diff}_+ \Sigma_0 \tag{5.3}$$

with $\mathrm{Ker}\,\Pi_* = \Gamma$. So we obtain $N(\Gamma)/\Gamma \cong \mathrm{Diff}_+ \Sigma_0$ as well as an isomorphism

$$\Pi_*: \quad Q_0(\Gamma) \to \mathrm{Diff}_0(\Sigma_0) \tag{5.4}$$

Any element $\omega_*$: $T(\Gamma) \to T(\Gamma)$ for $\omega \in N(\Gamma)$ is called an allowable map of $T(\Gamma)$ onto itself.

On the other hand, any orientation-preserving homeomorphism $h$ of $\Sigma_0$ onto itself also induces a nontrivial biholomorphic automorphism $h^*$ of $\mathscr{T}_{p,n} \cong T(\Gamma)$. This can be seen in the following way. Let us describe points of $\mathscr{T}_{p,n}$ as pairs $(\Sigma_i, f)$ (where $f$ is a quasiconformal map of the fixed Riemann surface $\Sigma_0$ onto a "variable" one $\Sigma_i$) with the equivalence relation

$(\Sigma_1, f_1) \sim (\Sigma_2, f_2)$ if $f_2 \circ f_1^{-1}$: $\Sigma_1 \to \Sigma_2$ is homotopic to a conformal map. The pair $(\Sigma_0, \text{id})$ is taken as the origin of $\mathscr{T}_{p,n}$. Now we have

$$h^*: \quad (\Sigma_i, f) \to (\Sigma_i, fh) \tag{5.5}$$

Transformations $h^*$ form a modular group $\text{Mod}_{p,n} \cong M(\Gamma)$, i.e., to each such $h^*$ these corresponds an appropriate element $\omega^*$ and vice versa. For $p \geqslant 3$ we have $\text{Mod}\,\Gamma = \Pi_0 \,\text{Diff}_+ \,\Sigma_0$ and $\text{Mod}\,\Gamma = \Pi_0 \,\text{Diff}_+ \,\Sigma_0/Z_2$ for $p = 2$. In other words $\text{Mod}\,\Gamma$ is isomorphic to the group of outer automorphisms $\text{Out}\,\Pi_1$ of $\Pi_1 \Sigma_0$ generated by Dehn twists and $\text{Out}\,\Pi_1/Z_2$, respectively.

Let $f$ be a quasiconformal map of a Riemann surface $\Sigma_1$ onto another $\Sigma_2$ and let $K(f)$ denote its global dilation. The Teichmüller metric on $\mathscr{T}_{p,n}$ is given by

$$\langle (\Sigma_1, f_1), (\Sigma_2, f_2) \rangle = \tfrac{1}{2} \log K(f) \tag{5.6}$$

where $f$ has a minimal dilatation in the homotopy class of $f_2 \circ f_1^{-1}$. Let us denote an element of $T(\Gamma) = \mathscr{T}_{p,n}$ by $\tau = (\Sigma_i, f)$ and let $\chi$ belong to the modular group $\text{Mod}\,\Gamma = \text{Mod}_{p,n}$. Bers (1978) considered the problem of minimalizing $\langle \tau, \chi(\tau) \rangle \equiv \langle \tau, h^*(\tau) \rangle$ for some self-mapping $h$ of $\Sigma_0$ by varying the conformal structure $\Sigma_i$ and by varying $h'$ in the isotopy class of $h$. If so, we call $\Sigma_0$ and $h$-minimal conformal structure and we call $f \cdot h \cdot f^{-1}$ an absolutely extremal self-mapping of the Riemann surface $f(\Sigma_0) = \Sigma_i$. In this case

$$K(f \cdot h \cdot f^{-1}) \leqslant K(f_i \cdot h \cdot f_i^{-1}) \qquad \forall f_i: \quad \Sigma_0 \to \Sigma_i \tag{5.7}$$

An element $\chi \equiv h^*$ of the modular group $M_{p,n}$ for which there exists $\tau \in \mathscr{T}_{p,n}$ such that the function $d(\tau)$ given by

$$d(\tau) = \langle \tau, h^*(\tau) \rangle \tag{5.8}$$

vanishes at $\tau$ is called elliptic (in this case $\tau$ is a fixed point of $h^*$: $\mathscr{T}_{p,n} \to \mathscr{T}_{p,n}$). If there exists an element $\tau$ such that

$$d(\tau) = \inf_{\tau \in \mathscr{T}_{p,n}} \langle \tau, h^*(\tau) \rangle = a > 0$$

then $h^*$ is called hyperbolic. So the function $d(\tau)$ given by (5.8) has an absolute minimum $a(h^*)$ (equal to zero or to $a > 0$) if $h^*$ is elliptic or hyperbolic. Bers has proven that in the latter case $h^*$ has to map some Teichmüller line onto itself [or equivalently that $K(\omega^2) = K(\omega)^2$, where $\omega \in N(\Gamma)$ and $\omega^* \cong h^*$]. $h^*$ has this property if and only if $h$ does not preserve any collection of admissible Jordan curves on a surface $\Sigma$.

A map $\chi \cong h^* \in \mathrm{Mod}_{p,n}$ is elliptic if and only if $h$ is isotopic to a periodic mapping. So if $h$ has infinite order and is irreducible, then it has to be hyperbolic, i.e., $h$ is an absolutely extremal self-mapping with dilatation $K > 1$. Thurston (n.d.) and Kra (1981) have shown independently the following, very important property of hyperbolic elements of the modular group $M(\Gamma) \cong \mathrm{Mod}_{p,n}$.

*Theorem 5.1.* Let $h$: $\Sigma_0 \to \Sigma_0$ be an absolutely extremal self-mapping with dilatation $K > 1$. Then $K$ is an algebraic integer.

## 5.2. Strings and Discreteness of the Symmetry Group

Let $q$ denote a unique quadratic differential on a compact Riemann surface $\Sigma_0$ determined by some concrete "observer" $e_0 \in \mathbb{R}^{1,D}$ and by a Lorentzian structure $\Sigma^L$ of the worldsheet. Let $q$ be a Jenkins–Strebel differential with only one cylinder of each type. Let $\alpha$ and $\beta$ be closed horizontal and vertical trajectories in each of the two homotopy classes, respectively. It is known (Kra, 1981) that in this case each component of $\Sigma_0 - \alpha \cup \beta$ is contracible. This implies that $\alpha \cup \beta$ intersects every admissible curve on $\Sigma_0$. Let $\gamma$ be an element of the Fuchsian group $\Gamma$ whose axis projects onto a filling curve $\alpha \cup \beta$. A hyperbolic element of $\Gamma$ which has this property is called essential (notice that $\Gamma$ consists of only hyperbolic and parabolic elements).

By the Dehn theorem any orientation-preserving homeomorphism of a Riemann surface $\Sigma_0$ onto itself is homotopic to the product of Dehn twists. Let $\tau_\alpha$ and $\tau_\beta$ denote the Dehn twists about $\alpha$ and $\beta$, respectively. Then the self-mapping $h$ of $\Sigma_0$ given by

$$h = \tau_\alpha \tau_\beta^{-1} \tag{5.9}$$

is irreducible. So it induces, according to Section 5.1, a map $h^*$ which corresponds to some allowable hyperbolic map $\omega$ of $U$.

Thus we have the following situation: If $q$ a Jenkis–Strebel quadratic differential with only one cylinder of each type and with $\alpha$, $\beta$ as core curves of these cylinders, respectively, then:

1. $\alpha \cup \beta$ is a filling curve.
2. A hyperbolic element of $\Gamma$ whose axis projects onto $\alpha \cup \beta$ is an essential element.
3. The map $h = \tau_\alpha \tau_\beta^{-1}$ is an absolutely extremal self-mapping of $\Sigma_0 \cong U/\Gamma$

By our assumption the quadratic differential $q$ of $\Sigma_0$ which determines the $P$-line is exactly of this type. Moreover, its properties described in the introduction imply that both its ring domains $R_1 = \Sigma_0 - \Gamma_q$ and $R_2 = \Sigma_0 - \Gamma_{-q}$ have the same heights and the same circumferences. Let us assume, for simplicity, that the heights $h_1$ and $h_2$ of $R_1$ and $R_2$, respectively, are equal to

$$h_1 = h_2 = 1 \tag{5.10}$$

and that the circumferences $c_1$ and $c_2$ satisfy

$$c_1 = c_2 = c \tag{5.11}$$

It is known (Masur, 1980) that the Teichmüller map from $\tau_0 = (\Sigma_0, \mathrm{id})$ to $h^*(\tau_0)$ is a quasiconformal map whose Beltrami differential is given by a quadratic differential $q_1$,

$$q_1 = \left(1 - \frac{(c^2+4)^{1/2} - c}{2}\, i\right) q \tag{5.12}$$

and its global dilatation $K$ is equal to

$$K = \frac{2 + c^2 + c(c^2+4)^{1/2}}{2} \tag{5.13}$$

As mentioned in Section 5.1, $K$ has to be algebraically integral (Thurston, n.d.; Kra, 1981). More precisely, $K$ is an eigenvalue of some integral matrix (since any holomorphic quadratic differential related to the $P$-line has to be a square of an Abelian differential of the first kind; see Section 2). So, since $K$ given by (5.13) has such a property, only a discrete set $\mathcal{O}$ of elements $e'$ from the unit timelike hyperboloid $H$ of $\mathbb{R}^{1,D}$ can "produce" $P$-lines related to the above quadratic differentials. So if we assume that we can relate to a string world sheet some physical object and if we agree that a Lorentzian structure of the world-sheet is important and that a Jenkins–Strebel differential related to the $P$-line with only one horizontal and vertical cylinder of the same heights and circumferences play roles in physics, then only a discrete set of "observers" can observe the same physical rules.

Moreover, if only a finite number of pieces of physical objects related to strings with compact sheets is present in nature, then we would have a cellular structure of our space-time (this is again a consequence of the fact that the circumference $c$ is determined by an eigenvalue of a matrix with integer entries).

## 6. *P*-LINES AND HAMILTONIAN SYSTEM

The Teichmüller space of a compact Riemann surface $\Sigma_0$ has many equivalent realizations. Some of them possess global coordinates, but some do not. In this section we will treat $\mathscr{T}_p$ as a Teichmüller space of marked hyperbolic surface. In other words, we will consider the space of hyperbolic surfaces together with a fixed isomorphism on $\Pi_1\Sigma$ to the Möbius group $\text{Möb}_{\mathbb{R}} \cong PSL(2, \mathbb{R})$ of isometries of the open disc; two surfaces are thought to be equivalent if there is an isometry between them respecting this isomorphism.

### 6.1. Symplectic Structure on Teichmüller Space

Let us start with a decomposition of each surface $\Sigma$ into so-called pairs of pants. For this let $\alpha_1, \ldots, \alpha_{2p-3}$ be a set of admissible Jordan curves on $\Sigma$. A collection of these $3p-3$ disjoint simple closed curves separate $\Sigma$ into $2p-2$ surfaces $S_1, \ldots, S_{2p-2}$ each of which is homeomorphic to $S^2$ minus three open discs. Each pair of pants $S_i$, $i=1, \ldots, 2p-2$, has a hyperbolic structure with geodesic boundary, i.e., each component of $\partial S_i$ is a geodesic simple lop. The lengths $l_j$, $j=1, \ldots, 3$, of the boundaries of $S_i$ may be arbitrarily prescribed in the interval $(0, \infty)$.

Let us fix one such partition, say $\mathscr{P}$, of a surface $\Sigma_0$ represented by $n=3p-3$ geodesics $\{\alpha_j\}_{j=1,\ldots,n}$. At each $\alpha_i$ we define two parameters:

1. The length $l_{\alpha_j}=l_j$ of the unique simple closed geodesic on $\Sigma_0$ freely homotopic to $\alpha_j$.
2. The hyperbolic displacement $\tau_j$ between canonical points on each side of $\alpha_j$.

(Let us recall that each boundary of pants has two canonical points: the endpoints of the length-minimizing geodesics connecting the other boundaries.) An orientation of $\Sigma_0$ gives the sign of the displacement $\tau_j$. So the partition $\mathscr{P}$ of $\Sigma_0$ allows us to define parameters $(l_j, \tau_j)$ for any element $\Sigma$ of $\mathscr{T}_p$. The value of $l_j$ at a marked surface $\Sigma \in \mathscr{T}_p$ is the length of the unique $\Sigma$-geodesic determined by $\alpha_j$, and the twist parameters $\tau_j$ are defined similarly as described above (Abikoff, 1980).

The coordinates $(l_j, \tau_j)$, $1 \leqslant j \leqslant n$, are called Fenchel–Nielsen coordinates for the Teichmüller space $\mathscr{T}_p$ of marked Riemann surfaces. They are global coordinates and they vary in the intervals $0 < l_j < \infty$, $-\infty < \tau_j < \infty$, i.e., $\mathscr{T}_p$ is a cell of real dimension $6p-6$. On $\mathscr{T}_p$ there also exists another (besides the Teichmüller one) metric. This is the so-called Weil–Peterson metric. It is Kählerian, has negative holomorphic curvature, and is not complete (Wolpert, 1975). The Weill–Petterson–Kähler form on $\mathscr{T}(\Sigma) \cong \mathscr{T}_p$ provides a symplectic structure $\omega$ on $\mathscr{T}(\Sigma)$ which, of course, does not depend on the initial partition $\mathscr{P}$ of $\Sigma_0$ into pairs of pants. The results of Wolpert (1985) tell us that if $(l_j, \tau_j)$ are global Fenchel–Nielsen coordinates given by some concrete partition $P$ of $\Sigma_0$, then the symplectic form $\omega$ can be written as

$$\omega = -\sum_{j=1}^{n} d\tau_j \wedge dl_j, \qquad n = 3p-3 \tag{6.1}$$

[We recall that the mapping given by global coordinates $(l_j, \tau_j)$

$$\mathscr{T}(\Sigma_0) \to \mathbb{R}_+^{3p-3} \times \mathbb{R}^{3p-3} \tag{6.2}$$

is a real analytic diffeomorphism (Abikoff, 1980).]

The fact that the Teichmüller space $\mathscr{T}_p \cong \mathscr{T}(\Sigma_0)$ can be considered as a symplectic manifold allows us to define the notion of its Lagrangian submanifolds. Wolpert's results suggest that each concrete partition $P$ on $\Sigma_0$ introduces two natural, mutually transverse Lagrangian foliations $\mathscr{F}_1$ and $\mathscr{F}_\tau$ of $(\mathscr{T}_p, \omega)$. The foliation $\mathscr{F}_\tau$ is formed by real submanifolds of $\mathscr{T}(\Sigma_0)$ which are determined by the conditions $\{\tau_i = \text{const}\}_{i=1,\ldots,n}$ and, similarly, the foliation $\mathscr{F}_1$ is given by the condition $\{l_i = \text{const}\}_{i=1,\ldots,n}$. If we think of a foliation $\mathscr{F}$ as an integrable distribution, i.e., subbundle $E \subset T\mathscr{T}_p$ of a tangent bundle of a Teichmüller space, then $\mathscr{F}$ is Lagrangian if and only if the fibers of $E$ are Lagrangian subspaces of the fibers of $T\mathscr{T}_p$. The distribution $E_1$, corresponding to $\mathscr{F}_1$, is spanned by Fenchel–Nielsen (FN) vector fields

$$E_1 = \left\{\frac{\partial}{\partial \tau_1}, \ldots, \frac{\partial}{\partial \tau_n}\right\} \tag{6.3}$$

and $\omega(\partial/\partial\tau_i, \partial/\partial\tau_j) = \partial l_j/\partial\tau_i = 0$ guarantees that $E_1$ is integrable. The distribution $E_\tau$ is spanned by vectors $\{\partial/\partial l_i\}_{i=1,\ldots,n}$ and is integrable as well. Using the Petersson series $\theta_i$ related to a closed loop determined by $\alpha_i$, $i = 1, \ldots, n$, we have (Wolpert, 1985)

$$\frac{\partial}{\partial \tau_i} \cong \frac{i}{\pi} (Jmz)^2 \bar{\theta}_{\alpha_i} \qquad \text{and} \qquad dl_i \cong \frac{2}{\pi} \theta_{\alpha_i} \tag{6.4}$$

Let us recall the following Weinstein (1971) theorem.

*Theorem 6.1.* Let $\mathscr{F}$ be a Lagrangian foliation of some symplectic manifold $(M, \omega)$. Let $N \subset M$ be a Lagrangian submanifold which is transverse to $\mathscr{F}$ in the sense that

$$TM|_N = TN \oplus E|_N$$

where $E \subset TM$ is a subbundle corresponding to $\mathscr{F}$. Then there is a diffeomorphism $f$: $(M, N) \to (T^*N, Z_N)$ such that $f|_N = I_N$ modulo identification of $N$ with $Z_N$; $f^*\omega_N = \omega$ and $f$ takes leaves of $\mathscr{F}$ onto the fibers of $T^*N$ ($\omega_N$ is the natural symplectic form on the cotangent bundle $T^*N$).

Let us fix the foliation $\mathscr{F}_\tau$, Its leaves are determined by constant values of $\tau_1 \cdots \tau_n$. Let $\mathscr{L}_0 \in \mathscr{F}_1$ be given by some concrete positive numbers $l_0 = (l_1, \ldots, l_n)$. Of course $\mathscr{L}_0$ is transverse to $\mathscr{F}_\tau$ and we can apply Theorem 1. In our case (since we have an FN analytic diffeomorphism between $\mathscr{T}_p$ and $\mathbb{R}_+^{3p-3} \times \mathbb{R}^{3p-3}$) there exists the corresponding map

$$f\colon\ \mathscr{T}_p \longrightarrow T^*\mathscr{L}_0 \tag{6.5}$$

and we have

$$\mathscr{F}_\tau \ni \mathscr{L}_\tau \xrightarrow{f} \Pi^{-1}(\tau)$$

where $\tau = (\tau_1, \ldots, \tau_n)$ parametrize the leaf $\mathscr{L}_0 \in \mathscr{F}_1$, and $\Pi$: $T^*\mathscr{L}_0 \to \mathscr{L}_0$ is the natural projection.

### 6.2. *P*-Lines and the Hamiltonian System

In this section we will consider $P$-lines which are related to Jenkis–Strebel differentials with $3p-3$ cylinders. Let us notice that if we assume that we have some fixed, concrete Fenchel–Nielsen coordinates on $\mathscr{T}_p$ given by the concrete partition $\mathscr{P} \equiv \{\alpha_1 \cdots \alpha_{3p-3}\}$, then we can introduce a field $\varphi(x)$ of (normed) holomorphic quadratic differentials on the leaf $\mathscr{L}_0 \in \mathscr{F}_1$ (i.e., $X \in \mathscr{L}_0 \subset \mathscr{T}_p$) uniquely determined by the following conditions:

1. For each $X \in \mathscr{L}_0$, $\varphi(X)$ is a Jenkins–Strebel differential with $3p-3$ cylinders.
2. The ring decomposition of $X$ given by the differential $\varphi(X)$ is related to the Jordan curves $\alpha_1, \ldots, \alpha_n$.
3. The circumferences of the corresponding cylinders determined by $\varphi(X)$ are equal to $l_1, \ldots, l_n$ for every $X \in \mathscr{L}_0 \subset \mathscr{T}_p$ [measured in the $\varphi(X)$ metric].

This last condition means that our field $\varphi(X)$ of quadratic differentials is uniquely determined by a partition $\mathscr{P}$ (and leaf $\mathscr{L}_0$).

*Lemma 6.1.* Let $\varphi(X)$ be a Jenkins–Strebel differential on $X \in \mathscr{T}_p$ with $3p-3$ cylinders related to $\{\alpha_i\}_{i=1,\ldots,n}$, $n = 3p-3$. The Poincaré lengths of the $[\alpha_i]$ along the Teichmüller geodesic $l_X$ determined by $\varphi(X)$ cannot be constant functions on any open interval of $l_X$.

*Proof.* For a given Teichmüller parameter $K$ let $\mathring{\mu}_K$ be the Beltrami differential tangent to $l_X$ at $X_K = l_X(K)$. Let $\{l_{\alpha i}, l_{\beta i}\}_{i=1,\dots,n}$ be local coordinates in some neighborhood $\mathscr{U}$ of $X_K \in \mathscr{T}_p$ related to corresponding Fenchel–Nielsen vector fields $t_{\alpha i}$, $t_{\beta i}$ (Wolpert, 1982). There exists a neighborhood $\mathscr{U}_0 \subset \mathscr{U}$ such that the Beltrami differential $\mathring{\mu}_K$ can be uniquely written in $\mathscr{U}_0$ as

$$\mathring{\mu}_K = \sum_{i=1}^{n} a_i(K) t_{\alpha i} + b_i(K) t_{\beta i} \tag{6.6}$$

where $a_i$, $b_i$ are real functions of $K$. Let us consider the $K$th model of $X_K$. We see that along $l_K$ circumferences as well as heights of corresponding cylinders have to vary for each $i = 1, \dots, n$. So the coefficients $a_i(K)$ as well as $b_i(K)$ cannot vanish in $\mathscr{U}_0$. Now we have

$$(\mathring{\mu}_K, dl_i) = (b_i(K) t_{\beta i}, dl_{\alpha i}) = b_i(K) \sum_{p \in \# \alpha_i \cap \beta_i} \cos \beta_p \neq 0 \tag{6.7}$$

which completes the proof.

*Corollary.* Let $L_X$ be the Teichmüller line determined as in Lemma 6.1. Let $(l_i, \tau_i)$ be Fenchel–Nielsen coordinates determined by the partition $\mathscr{P} \equiv \{\alpha_1 \cdots \alpha_n\}$. Let $\mathscr{L}_0$ be a Lagrangian submanifold of $(\tau_p, \omega)$ (where $\omega$ is the Weil–Peterson–Kähler symplectic form) given by $l \equiv (l_1, \dots, l_n) = \text{const}$. Then the Teichmüller geodesic $l_X$ parametrized as

$$l_X(K) = (l_1(K), \dots, l_n(K), \tau_1(K), \dots, \tau_n(K)) \tag{6.8}$$

cannot lie in $\mathscr{L}_0$.

*Lemma 6.2.* Let $\varphi(X)$ be a field of Jenkis–Strebel differentials on a Lagrangian submanifold $\mathscr{L}_0$ uniquely introduced by conditions 1–3. The set of Teichmüller geodesics determined by $\varphi(X)$ form a one-dimensional fiber space over $\mathscr{L}_0$.

*Proof.* From Lemma 6.1 and from the corollary any geodesic $l_X \cong \varphi(X)$ cannot lie in $\mathscr{L}_0$ and any vector tangent to $l_X(K)$ cannot be tangent to a leaf of $\mathscr{F}_1$ through $l_X(K)$. So, the only thing we have to show is that the geodesic $l_X$ cannot return to $\mathscr{L}_0$. To see this, let us notice that if it returns to $\mathscr{L}_0$, then there has to exist a point $L_X(K) = y \in \mathscr{L}_1 \in \mathscr{F}_1$, $l \neq l_0$, on the geodesic $l_X$ such that the vector tangent to $l_X$ and $y$ has to be tangent to $\mathscr{L}_1$. However, this means that this vector would be related to a pure Fenchel–Nielsen twist $\tau = (\tau_1, \dots, \tau_n)$, which, as we know, is impossible.

Now, if two lines $l_X$ and $l_{X'}$ given by $\varphi(X)$ and $\varphi(X')$, respectively, $X$, $X' \in \mathscr{L}_0$, $X \neq X'$, would cross each other, then of course the line $l_X$ has a return back to $\mathscr{L}_0$ by the uniqueness of the Teichmüller map. This completes the proof.

*Lemma 6.3.* Let $(l_1, \ldots, l_N, \tau_1, \ldots, \tau_N)$ be Fenchel–Nielsen coordinates introduced by the partition $\mathscr{P} \equiv \{\alpha_1, \ldots, \alpha_n\}$. Let $\forall X \in \mathscr{L}_0$, $l_X \equiv (l_1(K), \ldots, l_n(K), \tau_1(K), \ldots, \tau_n(K))$ be the Teichmüller geodesic determined by $\varphi(K)$ as in Lemma 6.2. Let $\Pi_\tau$ map the geodesic $l_X$ through $X \in \mathscr{L}_0$ into $\Pi_\tau(l_X) = (l_1(K), \ldots, l_n(K), \tau_1, \tau_n)$, where $X \equiv (l_1, \ldots, l_n, \tau_1, \ldots, \tau_n)$. Curves $\Pi_\tau(l_x)$ are the integral curves of some Hamiltonian system on $\mathscr{T}_p$.

*Proof.* We can say that $\Pi_\tau l_X$ is a projection of $l_X$ into a Lagrangian submanifold $f^*(T^*_{(1,\tau)}\mathscr{L}_\tau$ [here $f$ is given by (6.5)]. Let us denote $\Pi_l l_x$ by $\zeta_X$. So along any curve $\zeta_X$. $X \in \mathscr{L}_0$, we have $\partial\tau_i/\partial K = 0$. Lines $\zeta_X$ will form integral curves of some Hamiltonian system on $\mathscr{T}_p$ if we find a function $H$ on $\mathscr{T}_p$ such that

$$\frac{\partial H}{\partial l_i} = 0 \qquad \text{and} \qquad \frac{\partial l_i}{\partial K} = \frac{\partial H}{\partial \tau_i}$$

along each $\zeta_X$. Let us recall that $l_i$ as a function on the Teichmüller space $\mathscr{T}_p$ determines the section $dl_i$ of the cotangent bundle over $\mathscr{T}_p$. By Gardiner's (1987) formula we know that for any Beltrami differential $\nu$ of compact surface Fuchsian group $\Gamma$ the differential $dl_i$ evaluated on the tangent vector $\nu$ is

$$(\nu, dl_i) = \frac{2}{\pi} \operatorname{Re} \int_\Delta \nu\theta_i \tag{6.9}$$

where $\Delta$ is a measurable fundamental domain for $\Gamma$ and $\theta_i$ is the Petersson series of a simple closed geodesic $[\alpha_i]$ on $X \equiv \mathscr{U}/\Gamma$.

Let $\mu_K^X$ be the Beltrami differential tangent to a line $l_X \equiv \varphi(X)$ at the point $l_X(K) \equiv X_K$. If $z_K$ denotes the natural parameter of the terminal quadratic differential on $X_K$ [determined by $\varphi(X)$], then

$$\mu_K^X = \frac{1}{2\pi} \frac{d\bar{z}_K}{dz_K} \tag{6.10}$$

The Petersson series $\theta_i$ determines a quadratic differential on $X_K$ which in the natural parameter $z_K$ has a form $\tilde{\theta}_i \cdot dz_K^2$. So $dl_i/dK$ along $l_X$ can be written as

$$\frac{dl_i}{dK} = \frac{1}{\pi K} \operatorname{Re} \sum_j \int_{R_j} \tilde{\theta}_i \, dz_K \, d\bar{z}_K \tag{6.11}$$

where $R_i$ is an appropriate ring domain on $X_K$. If the expression on the right side of (6.11) is a function (in the worst case) of $K$ and $\tau_i$ only, then our curves $\zeta_X$ will be integral curves of some Hamiltonian system on $\mathcal{T}_p$ (this condition is of course not necessary). In this case our "time" parameter will be equal to the Teichmüller parameter $K$ along the Teichmüller geodesic.

Since $l=(l_1, \ldots, l_n)$ is constant on $\mathcal{L}_0$, we will denote each Teichmüller geodesic $l_X \cong \varphi(X)$ by $l_\tau$. We will show that $\partial l_i/\partial K$ along $l_\tau$, $\tau=(\tau_1, \ldots, \tau_n)$, is equal to $\partial l_i/\partial K$ along $l_{\tau'}=(\tau'_1, \ldots, \tau'_n)$ for each $\tau, \tau' \in \mathcal{L}_0$ with $\tau_i=\tau'_i$. In other words, $\partial l_i/\partial K$ is the same along the Teichmüller geodesic starting from those elements $X \in \mathcal{L}_0$ that have the same $i$th coordinate $\tau_i$. To see this, let us recall how we have constructed our field $\varphi(X)$. Namely, the set of simple, closed curves $\{\alpha_i\}_{i=1,\ldots,n}$ determines the partition of $X$ onto $2p-2$ pairs of pants. The quadratic differential $\varphi(X)$ determines the decomposition of $X$ onto $N=3p-3$ ring domains related to the same set of curves $\{\alpha_i\}_{i=1,\ldots,n}$. The length of the circumference of a given ring domain is equal to the Poincaré length of a unique hyperbolic geodesic in the same homotopy class as this circumference. Since Poincaré lengths are constant on $\mathcal{L}_0$, the ring decomposition of $X$ and $X'$, $X \neq X'$, $X, X' \in \mathcal{L}_0$, will contain cylinders with the same appropriate circumferences but with different heights. Now, from the construction of an arbitrary hyperbolic surface by assembling pairs of pants, we see that all elements of $\mathcal{L}_0$ with the same $i$th $\tau$ coordinate $\tau_i$ will have the same height $h_i$ of the corresponding ring $R_i(X)$. So when we take $K$-models for $l_X(K)=l_\tau(K)$ with $\tau_i=\text{const}$ we see that all rings $R_i(l_\tau(K))$ for a fixed $K$ are conformally equivalent to each other. This means that the relation between the Poincaré length $l_i$ and the Teichmüller parameter $K$ is, for all geodesics $l_\tau$ with $\tau_i=\text{const}$, the same. Hence, the right sdide of (6.11) can be written as

$$\frac{1}{\pi K} \operatorname{Re} \sum_j \int_{R_j} \tilde{\theta}_i \, dz_K \, d\bar{z}_K = \frac{1}{\pi K} f_i(K, \tau_i) \tag{6.12}$$

Thus, the lines $\zeta_X(K)$ form integral curves of a Hamiltonian system with a Hamiltonian equal to

$$H = H(K, \tau_1, \ldots, \tau_n) = \sum_i \int \frac{1}{\pi K} f_i(K, \tau_i) \, d\tau_i$$

In other words, the curves $\zeta_X(K)$ satisfy

$$\frac{d\tau_i}{dK} = 0 = \frac{\partial H}{\partial l_i}$$

$$\frac{dl_i}{dK} = \frac{1}{\pi K} f_i(K, \tau_i) = \frac{\partial H}{\partial \tau_i}$$

which completes the proof.

We can see the equality (6.12) also using the Wolpert result. Namely we have the following result.

*Proposition 6.1* (Wolpert, 1981). Let $f_t$: $X_0 \to X_t$ be a smooth deformation. Denote by $g$ the hyperbolic line element on $X_t$. Choose a closed curve $\alpha$ on $X_0$ and denote by $l(t)$ the length of the unique $g$-geodesic freely homotopic to $f_t(\alpha)$ on $X_t$. Then

$$\frac{d}{dt} l(t) = \int_{\alpha_0} \frac{d}{dt} f_t^* g$$

where $\alpha_0$ is the $g$-geodesic on $X_0$.

Now, the differential $\varphi$ determines the flat metric $ds^2 = |dz|^2$ on each cylinder $R_j$, $j = 1, \ldots, n$ on $X$. Let the Teichmüller map $f_K$: $X \to X_K$ be of a form

$$f_K\colon \quad x_1 \to x_1' = Kx_1$$
$$x_2 \to x_2' = x_2$$

where $z = x_i + ix_2$ and $\xi = x_1' + ix_2'$ are holomorphic, natural coordinates for the appropriate cylinders on $X$ and $X_K$, respectively.

Let $y_1$, $y_2$ be local coordinates on $X$ such that

$$\cosh^2 y_2 \, dy_1^2 + dy_2^2 = \lambda(x_1, x_2)\,(dx_1^2 + dx_2^2) \tag{6.13}$$

is the hyperbolic metric on $X$ and let $y_1'$, $y_2'$ be the same for $X_K$, i.e.,

$$\cosh^2 y_2' \, dy_1'^2 + dy_2'^2 = \lambda'(x_1', x_2')\,(dx_1'^2 + dx_2'^2) \tag{6.13'}$$

is the unique hyperbolic metric on $X_K$. Let us notice that (when $\tau_i = \text{const}$) the functions $\chi_r(x_1, x_2) = y_r$ and $\chi_r'(x_1', x_2') = y_r'$, $r = 1, 2$, will have the same form in the variables $\{x_1, x_2\}$ and $\{x_1', x_2'\}$, respectively, i.e., $\chi_r'(x_1', x_2') = \chi_r(Kx_1, x_2)$. On $[\alpha_i]$, $\chi_2(x_1, x_2) = 0$, and on $[\alpha_i]_K$, $\chi_2(Kx_1, x_2) = 0$. (Here $[\alpha_i]$ and $[\alpha_i]_K$ denote Poincaré geodesics in the homotopy class of $\alpha_i$ and $f_K(\alpha_i)$, respectively.) Now the metric $f_K^* g$ (from the Wolpert proposition) can be written as

$$\begin{aligned} g_K(x_1, x_2) &= f_K^*(\cosh^2 y_2' \, dy_1'^2 + dy_2'^2) \\ &= \cosh^2 f_2(Kx_1, x_2) \left( \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 \right)^2 \Bigg|_{(Kx_1, x_2)} \\ &\quad + \left( \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 \right)^2 \Bigg|_{(Kx_1, x_2)} \end{aligned}$$

By the Wolpert proposition

$$\frac{dl_i}{dK} = \int_{[\alpha_i]} \frac{\partial g_K}{\partial K} \tag{6.14}$$

and we see that the right side of (6.14) can depend only on $K$ and $l_i(1)$ such that (6.12) is satisfied.

By the properties of the Teichmüller maps, the functions $f_i$ are analytic and we can perform their continuations to the whole $\mathscr{T}_p$.

## 7. CONCLUSION

In Section 4 we saw that $P$-lines which are related to Jenkis–Strebel differentials (for example, "harmonic" $P$-lines) describe a world sheet which has to be created and which has to decay. In Bugajska (1990), 1991) we obtained that $P$-line satisfying the $P$-condition is "associated" to reductions of appropriate holomorphic $SL(2, \mathbb{C})$ bundles (over Riemann surfaces determined by this line) to the $SU(2)$ group. This means (Bugajska, 1990, 1991) that we have to deal with $SU(2)$ bundles over Riemann surfaces equipped with a concrete connection $A$. If we interpret this connection as a gauge field of weak interaction (which is responsible for a process of decay), then we see that these completely different approaches yield the same physical situation, namely decay and creation. Moreover, holomorphic quadratic differentials which satisfy the $P$-condition seem to be just Jenkins–Strebel differentials, or at least most of them (it is still an open question).

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